The $\mathfrak{M}_H(G)$-Conjecture and Mazur's conjecture for Abelian Varieties over Global Function Fields

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Abstract

This article is based on a talk presented at RIMS on non-commutative Iwasawa theory over global function fields. We begin by reviewing the origins of classical Iwasawa theory and its extension to the non-commutative setting for elliptic curves over number fields, as developed by Coates, Fukaya, Kato, Sujatha, and Venjakob.

We then turn to non-commutative Iwasawa theory for abelian varieties over a global function field F of characteristic different from p. For certain p-adic Lie extensions F_∞ containing the cyclotomic $\mathbb{Z}_p$-extension F^{cyc}, we establish the $\mathfrak{M}_H(G)$-conjecture for the Pontryagin dual of the Selmer group, providing a crucial structural result towards a non-commutative main conjecture. A key consequence of our essentially cohomological methods is the vanishing of the relevant Iwasawa μ-invariants. This result settles Mazur's conjecture in the function field setting.

Finally, we provide explicit computations comparing the generalised Euler characteristics and Akashi series of the relevant Selmer groups.

Contents

1 Introduction

Iwasawa theory provides a powerful framework for exploring the deep connections between arithmetic objects, such as ideal class groups and Selmer groups, and analytic objects, such as complex and p-adic L-functions. The theory was motivated by classical results relating the p-part of the ideal class group of a number field to special values of the Riemann zeta function. Building on these insights, Iwasawa developed a theory which studies the growth of arithmetic invariants in infinite towers of number fields.

Today, Iwasawa theory plays a central role in modern number theory, particularly in the study of elliptic curves and major conjectures, including the Birch–Swinnerton-Dyer conjecture. In what follows, we begin by revisiting its classical origins in the work of Kummer, then turn to its non-commutative developments due to Coates, Fukaya, Kato, Sujatha, and Venjakob, and finally consider the case of global function fields.

This is the setting of our recent work [7], from which we survey several of our main theorems. For clarity of exposition, some theorems are presented here in a more accessible formulation. The complete arguments, along with further results, can be found in [7].

2 Classical Iwasawa Theory and its Origins

The origins of Iwasawa theory lie in the mid-19th century work of Ernst Kummer on Fermat's Last Theorem. Kummer's strategy was to study the arithmetic of cyclotomic fields $\mathbb{Q}(\zeta_p)$, obtained by adjoining a primitive p-th root of unity to $\mathbb{Q}$. During this work, he uncovered a profound connection between the structure of their ideal class groups, $\mathrm{Cl}_{\mathbb{Q}(\zeta_p)}$, and the values of the Riemann zeta function at negative odd integers.

Kummer observed that if the ring of integers of $\mathbb{Q}(\zeta_p)$ were a unique factorisation domain (meaning its class number is one), the proof would be straightforward. His crucial insight, however, was to establish that a less restrictive condition is sufficient: the Fermat equation $x^p + y^p = z^p$ has no non-trivial integral solutions if p does not divide the class number $h_{\mathbb{Q}(\zeta_p)} = \#\mathrm{Cl}_{\mathbb{Q}(\zeta_p)}$. A prime p satisfying this condition is called *regular*. Note that proving Fermat's Last Theorem reduces to considering the exponents 4 and all odd primes, and Kummer's work successfully settled the theorem for all regular primes.

While today Fermat's Last Theorem is known to be true for all exponents thanks to the work of Wiles, Kummer's criterion for regularity remains a beautiful and foundational result. His theorem not only provides the conditional proof of Fermat's Last Theorem but also gives a concrete method for testing whether a prime is regular.

Theorem 2.1 (Kummer, 1850)**.** *Let p be an odd prime, and let $h_{\mathbb{Q}(\zeta_p)}$ denote the class number of the pth cyclotomic field $\mathbb{Q}(\zeta_p)$.*

(1) If $p \nmid h_{\mathbb{Q}(\zeta_p)}$, then the Fermat equation $x^p + y^p = z^p$ has no non-trivial integral solutions.

(2) We have $p \mid h_{\mathbb{Q}(\zeta_p)}$ if and only if p divides the numerator of at least one of the values

$$\zeta(-1) = -\frac{1}{12}, \quad \zeta(-3) = \frac{1}{120}, \quad \zeta(-5) = -\frac{1}{252}, \ldots, \zeta(4-p).$$

Equivalently, since $\zeta(1-2k) = -\frac{B_{2k}}{2k}$, where B_{2k} denotes the $2k$th Bernoulli number, $p \mid h_{\mathbb{Q}(\zeta_p)}$ if and only if p divides the numerator of B_{2k} for some k with $1 \leq k \leq \frac{p-3}{2}$.

This mysterious link between the p-part of the ideal class group of $\mathbb{Q}(\zeta_p)$ and the p-divisibility of special values of the zeta function was the primary inspiration for Kenkichi Iwasawa. He sought to understand this relation more deeply, and it became the starting point of what is now known as Iwasawa theory.

Iwasawa's idea was to study, instead of the p-part of the ideal class group $\mathrm{Cl}_{\mathbb{Q}(\xi_p)}$ in isolation, the behaviour and growth of the p-primary part of the ideal class groups over an infinite tower of number fields.

Fix a prime p, and consider the infinite cyclotomic extension

$$\mathbb{Q}(\xi_{p^\infty}) := \bigcup_{n=1}^{\infty} \mathbb{Q}(\xi_{p^n}).$$

This forms a $\mathbb{Z}_p^\times$-extension of $\mathbb{Q}$ with $\mathrm{Gal}(\mathbb{Q}(\xi_{p^\infty})/\mathbb{Q}) = \Delta \times \Gamma$, where Δ is cyclic of order 2 or $p-1$ according as $p = 2$ or p is odd, and Γ is non-canonically isomorphic to $\mathbb{Z}_p$. The unique $\mathbb{Z}_p$-extension of $\mathbb{Q}$ contained in $\mathbb{Q}(\xi_{p^\infty})$ is called the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, and is denoted by $\mathbb{Q}^{\mathrm{cyc}}$.

$$\begin{array}{c} \left.\begin{array}{c} \mathbb{Q}(\xi_{p^\infty}) \\ | \\ \mathbb{Q}(\xi_{p^n}) \\ | \\ \vdots \\ | \\ \mathbb{Q}(\xi_{p^2}) \\ | \end{array}\right\} \quad \Gamma \\ \left.\begin{array}{c} \mathbb{Q}(\xi_p) \\ | \\ \mathbb{Q} \end{array}\right\} \quad \Delta \end{array}$$

Iwasawa's strategy was to assemble the p-primary part $A_n := \mathrm{Cl}_{\mathbb{Q}(\xi_{p^n})}(p)$ of the class groups from each layer of this tower into a single, unified object by taking the direct limit $A_\infty = \varinjlim A_n$ and then the Pontryagin dual (the group of continuous homomorphisms into $\mathbb{Q}_p/\mathbb{Z}_p$) to turn the discrete infinite torsion group into a compact one. This leads to a seemingly more complicated arithmetic object. Remarkably, this object comes equipped with a rich algebraic structure: it is a module over an Iwasawa algebra. Given any profinite group G, the *Iwasawa algebra* $\Lambda(G)$ of G with coefficients in $\mathbb{Z}_p$ is defined by

$$\Lambda(G) = \varprojlim \mathbb{Z}_p[G/U],$$

where U runs over open normal subgroups, and the inverse limit is taken with respect to the natural projection maps. In this specific case, because $\Gamma \simeq \mathbb{Z}_p$, the algebra $\Lambda(\Gamma)$ is isomorphic to the ring of formal power series $\mathbb{Z}_p[[T]]$ via taking a topological generator γ of Γ to $1+T$. We shall use such an isomorphism to identify $\Lambda(\Gamma)$ with $\mathbb{Z}_p[[T]]$.

The structure theory for finitely generated $\Lambda(\Gamma)$-modules says, given such a module M, M is pseudo-isomorphic to a direct sum of a standard form:

$$M \sim \Lambda(\Gamma)^r \oplus \bigoplus_{i=1}^{s} \frac{\Lambda(\Gamma)}{(p^{\mu_i})} \oplus \bigoplus_{j=1}^{t} \frac{\Lambda(\Gamma)}{(f_j(T)^{k_j})}. \tag{1}$$

Here, $r \geq 0$ is the rank, the $\mu_i > 0$ are integers, and the $f_j(T)$ are distinguished polynomials (monic polynomials where the coefficients of the non-leading terms are divisible by p). The integers $\mu_i > 0$ define the μ-invariant, $\mu = \sum_{i=1}^{s} \mu_i$ of M, and the degrees of these polynomials f_j determine its λ-invariant, $\lambda = \sum_{j=1}^{t} k_j \deg(f_j)$. In the case where M is known to be torsion, meaning its rank r is zero, we can define the *characteristic ideal* to be the principal ideal $\mathrm{char}(M) = (p^{\mu} \prod_{j=1}^{t} f_j(T)^{k_j})$ generated by the torsion part of the decomposition. A generator of this ideal, denoted $f_M(T)$, is called a *characteristic power series* of M. This is defined up to multiplication by a unit in $\Lambda(\Gamma)$. These are central invariants in commutative Iwasawa theory.

3 Iwasawa theory of Elliptic Curves

The principles of Iwasawa theory were later extended to the setting of elliptic curves. In this context, the central arithmetic object of interest is the Selmer group, which can be viewed as an analogue of the ideal class group.

Let E be an elliptic curve over a number field F. For any algebraic extension L/F, the p-primary Selmer group of E over L is a subgroup of the Galois cohomology group $H^1(L, E_{p^\infty})$ which is the kernel of the global-to-local restriction map:

$$\mathrm{Sel}(E/L) := \ker\left(H^1(L, E_{p^\infty}) \to \prod_{w} H^1(L_w, E) \right). \tag{2}$$

Here, the product is taken over all places w of the field, and L_w denotes the completion of L at the place w. The Galois module E_{p^∞} is the group of all p-power torsion points of E (defined over a fixed algebraic closure of F). We note that when L is an infinite extension, the completion L_w is formed by taking the union of the completions of all finite extensions of F contained within L.

In Iwasawa theory, one studies these groups in the layers of a $\mathbb{Z}_p$-extension. Let $F^{\mathrm{cyc}} = F\mathbb{Q}^{\mathrm{cyc}}$ denote the cyclotomic $\mathbb{Z}_p$-extension of F, and write F_n for its n-th layer, where $\mathrm{Gal}(F_n/F) \simeq \mathbb{Z}/p^n\mathbb{Z}$. The Pontryagin dual of the direct limit of these groups, denoted $X(E/F^{\mathrm{cyc}})$, forms a module over the Iwasawa algebra $\Lambda(\Gamma)$, where $\Gamma = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$.

It is generally expected that it enjoys nice module-theoretical properties, namely Mazur's conjecture [16]:

Conjecture 1 (Mazur). *Let E be an elliptic curve over a number field F with good ordinary reduction at all primes above p. Then the Iwasawa module $X(E/F^{\mathrm{cyc}})$ is a finitely generated torsion $\Lambda(\Gamma)$-module.*

It is well known to be finitely generated, and establishing the torsionness is a foundational step toward the Iwasawa main conjecture, which says that the characteristic ideal of the module $X(E/F^{\mathrm{cyc}})$ is generated by a p-adic L-function that interpolates special values of the complex L-function of E. Proving the main conjecture allows one to relate the algebraic order of the Selmer group to analytic L-values, offering a powerful approach to the Birch–Swinnerton-Dyer conjecture.

Remark 3.1. *Mazur's conjecture is known in many cases. Notably:*

- *When $E/\mathbb{Q}$ and $F/\mathbb{Q}$ are abelian extensions (a consequence of Kato's work [12]).*
- *When $\mathrm{Sel}(E/F)$ is finite (a result of Mazur himself [16]).*

In view of the structure theory for $\Lambda(\Gamma)$-module (1), Mazur's conjecture can be reformulated as follows:

$$\text{Mazur's conjecture for } X(E/F^{\mathrm{cyc}}) \Leftrightarrow \frac{X(E/F^{\mathrm{cyc}})}{X(E/F^{\mathrm{cyc}})(p)} \text{ is a finitely generated } \mathbb{Z}_p\text{-module,}$$

where $X(E/F^{\mathrm{cyc}})(p)$ denotes the p-primary submodule. If this holds, one can define the characteristic ideal $\mathrm{char}(X(E/F^{\mathrm{cyc}}))$. The exact power of p dividing $\mathrm{char}(X(E/F^{\mathrm{cyc}}))$ is the μ-invariant of $X(E/F^{\mathrm{cyc}})$. A central conjecture, analogous to the classical case, predicts that this μ-invariant is zero. If the μ-invariant is zero, then $X(E/F^{\mathrm{cyc}})$ is not only a finitely generated torsion Λ-module but also a finitely generated $\mathbb{Z}_p$-module.

4 The Non-commutative Framework and the $\mathfrak{M}_H(G)$-Conjecture

Coates, Fukaya, Kato, Sujatha and Venjakob generalised the theory to p-adic Lie extensions F_∞ which contain F^{cyc}. For the discussions in the number field setting, I will let p denote an odd prime. Let F be a number field, and let E/F be an elliptic curve with good ordinary reduction at p. We can define the p-primary Selmer group $\mathrm{Sel}(E/F_\infty)$ of

E over F_∞, and we write the following for the compact Pontryagin dual of the p-primary Selmer group.

$$X(E/F_\infty) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Sel}(E/F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$$

A natural question to ask is: To what extent, and in what way, does the arithmetic of E over F^{cyc} relate to that of E over F_∞?

Let $G = \mathrm{Gal}(F_\infty/F)$, $\Gamma = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$ and $H = \mathrm{Gal}(F_\infty/F^{\mathrm{cyc}})$. Motivated by the above characterisation, Coates, Fukaya, Kato, Sujatha, and Venjakob introduced the following category of Iwasawa modules.

Let $\mathfrak{M}_H(G)$ denote the category of all finitely generated $\Lambda(G)$-modules M such that $M/M(p)$ is a finitely generated $\Lambda(H)$-module. In this setting, Coates, Fukaya, Kato, Sujatha, and Venjakob conjectured that the category $\mathfrak{M}_H(G)$ contains all torsion $\Lambda(G)$-modules of arithmetic interest. More precisely, they posed the following conjecture, now known as the $\mathfrak{M}_H(G)$-conjecture [5]:

Conjecture 2 ($\mathfrak{M}_H(G)$-conjecture)**.** *The Iwasawa module $X(E/F_\infty)$ lies in the category $\mathfrak{M}_H(G)$.*

This is a key conjecture which allowed them to attach a characteristic element to the Iwasawa module. This is essential for them to then formulate an Iwasawa main conjecture in the non-commutative setting. Given any module M in the category $\mathfrak{M}_H(G)$, one can associate a generalisation of the characteristic power series known as the Akashi series $\mathrm{Ak}(M)$ of M (see Section 7 (6)).

The $\mathfrak{M}_H(G)$-conjecture is still wide open, though there are known cases under $\mu = 0$ conditions [5]. In the next section, we make some observations in the case where F is a global function field and F_∞ satisfies some amiable conditions.

5 The Function Field Analogue and the Main Theorem

We now transition from number fields to the setting of global function fields. While Conjectures 1 and 2 were stated for elliptic curves defined over number fields, they can also be formulated for any abelian variety defined over a global function field F, and for any prime p, including $p = 2$.

From now on, let p be any fixed prime. Let F be a global function field of characteristic $\ell \neq p$; that is, F is a finite extension of the field $\mathbb{F}_{\ell^r}(T)$ of rational functions in one variable over the finite field $\mathbb{F}_{\ell^r}$ with ℓ^r elements. We can similarly define the cyclotomic $\mathbb{Z}_p$-extension F^{cyc} of F, obtained by adjoining to F the unique $\mathbb{Z}_p$-constant field extension $\mathbb{F}_{p^\infty}$ of $\mathbb{F}$.

Let F_∞ denote a p-adic Lie extension of F, that is, a Galois extension over F whose Galois group $G = \mathrm{Gal}(F_\infty/F)$ is a p-adic Lie group of positive dimension. We shall call a p-adic Lie extension F_∞ of F *admissible* if it satisfies the following conditions:

(i) F_∞ is unramified outside a finite set of primes of F;

(ii) F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension F^{cyc} of F;

(iii) G has no element of order p.

Condition (iii) ensures that G has finite p-cohomological dimension which coincides with the dimension of G as a p-adic Lie group, by a fundamental result of Lazard [14] and Serre [21].

Our first main result is as follows:

Theorem 5.1. *Let p be any prime. Let A be an abelian variety over a global function field F with characteristic prime to p, and let F_∞/F be an admissible p-adic Lie extension. Then the $\mathfrak{M}_H(G)$-conjecture holds for the Pontryagin dual $X(A/F_\infty)$ of the Selmer group* $\mathrm{Sel}(A/F_\infty)$*. Furthermore, $X(A/F^{\mathrm{cyc}})$ has μ-invariant zero, and the generalised μ-invariant (in the sense of [9, 27, 3]) of $X(A/F_\infty)$ is zero. In particular, Mazur's conjecture holds for A/F.*

This settles Mazur's conjecture and the $\mathfrak{M}_H(G)$-conjecture in the function field setting. This stands in sharp contrast to the number field case where the μ-invariant may be positive (see [16, §10, Example 2], [8]). Our results build on and contribute to a significant body of work on Iwasawa theory over function fields (see, e.g., [3, 13, 18, 19, 20, 25, 26, 28]); a key feature of our approach is that it is essentially cohomological. We will give a sketch of the proof and introduce two related results, referring the reader to our main paper [7] for complete details.

6 A Cohomological Description of the Selmer Group

For the proof of the Theorem 5.1, the following alternative description of Selmer groups will be crucially used to study them as $\Lambda(G)$-modules.

Let S be a nonempty finite set of primes of F containing all the primes where the abelian variety A has bad reduction, and all the primes of F which ramify in F_∞.

Let F_S be the maximal extension of F, contained in a fixed separable closure $\bar{F}$, which is unramified outside S. By the choice of S, we know F_∞ is contained in F_S. For any extension K of F contained in F_S, we write $G_S(K)$ for the Galois group $\mathrm{Gal}(F_S/K)$. When L is a finite extension of F, we define each prime v of F

$$J_v(A/L) = \bigoplus_{w|v} H^1(L_w, A)(p),$$

where w runs over all primes of L lying above v. Since L_w has characteristic different from p, we have $H^1(L_w, A)(p) \simeq H^1(L_w, A_{p^\infty})$ by Kummer theory. We shall make this identification without any further mention.

Let $\bar{L}_w$ be a separable closure of L_w. By choosing an embedding $\bar{F} \subset \bar{L}_w$, we can view $\mathrm{Gal}(\bar{L}_w/L_w)$ as a subgroup of $\mathrm{Gal}(\bar{F}/L)$. We then have the following localisation map

$$\lambda_S(A/L) : H^1(G_S(L), A_{p^\infty}) \to \bigoplus_{v \in S} J_v(A/L) \tag{3}$$

induced by restriction.

For an infinite separable extension K of F contained in F_S, we define

$$J_v(A/K) = \varinjlim J_v(A/L),$$

where L runs over all finite extensions of F contained in K and the limit is taken with respect to the restriction maps. We similarly define the localisation map

$$\lambda_S(A/K) = \varinjlim \lambda_S(A/L).$$

We will use the following description of the Selmer groups, which can be proven in the same way as in the case of elliptic curves over number fields (see [4, Lemma 2.3]).

Proposition 6.1. *Let K be an extension of F contained in F_S. Then $\mathrm{Sel}(A/K)$ satisfies the exact sequence*

$$0 \to \mathrm{Sel}(A/K) \to H^1(G_S(K), A_{p^\infty}) \xrightarrow{\lambda_S(A/K)} \bigoplus_{v \in S} J_v(A/K).$$

In fact, a key technical result of our work is that the final localisation map in this sequence is surjective. This is established in [7] using Jannsen's spectral sequence [10, 11] and Nekovář's duality theorem [17, 15], and it provides a crucial input for the comparison formula of Akashi series presented in Theorem 7.3.

7 Generalised Invariants and Comparison Formulae

We now present two key results from our paper [7], which we state here for elliptic curves. These results, which will be given as Theorem 7.2 and Theorem 7.3, provide comparison formulae relating the generalised Euler characteristics and Akashi series for the extensions F_∞/F and F^{cyc}/F. (For the more general treatment of abelian varieties and the proofs, we refer the reader to [7].) To situate our work, we first recall foundational results by Sechi and Zerbes.

Let Y be a discrete p-primary G-module. We say that Y has finite G-Euler characteristic if the cohomology groups $H^i(G, Y)$ are finite for all $i \geq 0$. In this case, we define its Euler characteristic $\chi(G, Y)$ to be the alternating product of the cardinality of these cohomology groups:

$$\chi(G, Y) = \prod_{i \geq 0} \left| H^i(G, Y) \right|^{(-1)^i}.$$

The significance of the Euler characteristic in Iwasawa theory lies in its deep connection to the algebraic structure of the corresponding Iwasawa module. In the classical commutative setting where $G \simeq \Gamma$ and the dual module $M = Y^\vee$ is a finitely generated torsion $\Lambda(\Gamma)$-module with characteristic power series $f_M(T)$, then the Euler characteristic of Y is finite if and only if $f_M(0) \neq 0$. When this holds, we have the Euler characteristic formula:

$$\chi(G, Y) = |f_M(0)|_p^{-1}, \tag{4}$$

where $|\cdot|_p$ is the p-adic valuation of $\mathbb{Q}$, normalised so that $|p|_p = p^{-1}$. Analogously to the number field case [6, 29, 30, 31], the G-Euler characteristic of $\mathrm{Sel}(E/F_\infty)$ is closely related to the Γ-Euler characteristic of $\mathrm{Sel}(E/F^{\mathrm{cyc}})$, where $\Gamma = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$. Indeed, Gianluigi Sechi proved the following theorem in his PhD thesis [20].

Theorem 7.1 (Sechi). *Let F be a global function field with characteristic $\ell \geq 5$. Let $p \geq 5$ be a prime different from ℓ. Let E/F be an elliptic curve, and let $\mathcal{S}$ be the finite set of primes of F where E has bad, and not potentially good, reduction. Let $\mathfrak{F}_\infty = F(E_{p^\infty})$ and $\Xi = \mathrm{Gal}(\mathfrak{F}_\infty/F)$. If $\mathrm{Sel}(E/F)$ is finite, then the Ξ-Euler characteristic of $\mathrm{Sel}(E/\mathfrak{F}_\infty)$ is given by*

$$\chi(\Xi, \mathrm{Sel}(E/\mathfrak{F}_\infty)) = \chi(\Gamma, \mathrm{Sel}(E/F^{\mathrm{cyc}})) \times \prod_{v \in \mathcal{S}} |L_v(E, 1)|_p.$$

Here, $L_v(E, s)$ is the Euler factor of the complex L-function of E over F at v.

Subsequent work has built on Sechi's result (see, e.g., [26], [2]); however, these advances relied on the assumption that the Selmer group over the ground field F is finite.

When $E(F)$ is infinite, $\mathrm{Sel}(E/F_\infty)$ does not have finite G-Euler characteristic. To address this, we introduce the generalised G-Euler characteristics, following the framework developed by Zerbes for number fields [30]. This construction requires the admissibility assumption (ii) that F^{cyc} is contained in F_∞. Let us denote $H = \mathrm{Gal}(F_\infty/F^{\mathrm{cyc}})$. For the computation and comparison of these generalised Euler characteristics, we introduce the following condition:

Hypothesis (H). $H^i(H, E_{p^\infty}(F_\infty))$ is finite for all $i \geq 0$.

Alternatively, we could use the hypothesis that $H^i(\mathcal{U}, E_{p^\infty}(F_\infty))$ is finite for all $i \geq 0$, where $\mathcal{U}$ is any open normal subgroup of H.

We remark that the condition (**H**) is satisfied by a wide class of p-adic Lie extensions, for example, when one of the following conditions is satisfied:

(1) $F_\infty = F(B_{p^\infty})$ where B is an abelian variety over F;

(2) $\mathrm{Lie}(H)$ is reductive.

Let W be a discrete p-primary Γ-module. Then we have

$$H^0(\Gamma, W) = W^\Gamma \quad H^1(\Gamma, W) = W_\Gamma,$$

and hence there is a map

$$\phi_W : H^0(\Gamma, W) \to H^1(\Gamma, W), \quad f \mapsto \text{ residue class of } f.$$

For a discrete p-primary G-module Y, we define

$$d^0 : H^0(G, Y) = H^0(\Gamma, Y^H) \longrightarrow H^1(\Gamma, Y^H) \hookrightarrow H^1(G, Y),$$

where the middle map is ϕ_{Y^H} and the last map is given by inflation. Similarly, for $i \geq 1$, define

$$d^i : H^i(G, Y) \twoheadrightarrow H^0(\Gamma, H^i(H, Y)) \longrightarrow H^1(\Gamma, H^i(H, Y)) \hookrightarrow H^{i+1}(G, Y),$$

where the first map is given by restriction and the middle map is equal to $\phi_{H^i(H,Y)}$. Note that the first map is surjective since $\mathrm{cd}_p(\Gamma) = 1$. We define d^{-1} to be the zero map.

For all $i \geq 0$, $(H^\bullet(G, Y), d^\bullet)$ forms a cocomplex. Denote its cohomology by $\mathfrak{H}^i$.

We say that the G-module Y has finite *generalised G-Euler characteristic* if $\mathfrak{H}^i$ is finite for all $i \geq 0$. In this case, the generalised Euler characteristic is defined as the alternating product

$$\chi(G, Y) = \prod_{i \geq 0} |\mathfrak{H}^i|^{(-1)^i}.$$

Let S be a nonempty finite set of primes of F containing all the primes where E has bad reduction, and all the primes of F which ramify in F_∞. We define S' to be the subset of primes of S whose inertia group in G is infinite. We can now state the following result:

Theorem 7.2. *Let p be any prime, and let F be a global function field with $\mathrm{char}(F) \neq p$. Let F_∞ be an admissible p-adic Lie extension of F with Galois group $G = \mathrm{Gal}(F_\infty/F)$. Let E be an elliptic curve defined over F. Assume that $\text{Ш}(E/F)(p)$ is finite and the condition* (**H**) *holds. Then $\mathrm{Sel}(E/F_\infty)$ has finite generalised G-Euler characteristic if and only if $\mathrm{Sel}(E/F^{\mathrm{cyc}})$ has finite generalised Γ-Euler characteristic. Moreover, if this is the case, then*

$$\chi(G, \mathrm{Sel}(E/F_\infty)) = \chi(\Gamma, \mathrm{Sel}(E/F^{\mathrm{cyc}})) \times \prod_{v \in S'} |L_v(E, 1)|_p. \tag{5}$$

Here, for a prime $v \in S'$, we denote by $L_v(E, s)$ the Euler factor of L–function of E at v

This theorem establishes the function field analogue of a result proven by Zerbes for number fields in her PhD thesis. Our result may thus be viewed as a “fibre-product” of the works of Sechi and Zerbes, both of whom conducted their doctoral research under the supervision of John Coates.

The importance of this generalised Euler characteristic lies in its relation with Akashi series introduced in [6], whose definition we now recall.

Denote by $Q(\Gamma)$ the fractional field of $\Lambda(\Gamma)$. To each M in $\mathfrak{M}_H(G)$, we attach a non-zero element $\mathrm{Ak}(M)$ of $Q(\Gamma)$. It can be shown that the homology groups $H_i(H, M)$ are finitely generated torsion $\Lambda(\Gamma)$-modules. Let $f_{M,i} \in \Lambda(\Gamma)$ be a characteristic power series for $H_i(H, M)$, and define

$$\mathrm{Ak}(M) = \prod_{i\geq 0} f_{M,i}^{(-1)^i}. \tag{6}$$

This product is finite because $H_i(H, M) = 0$ for $i \geq \dim(H)$, and it is well defined up to multiplication by a unit in $\Lambda(\Gamma)$, because each $f_{M,i}$ is. We call $\mathrm{Ak}(M)$ the *Akashi series*[1] of M. It generalises the notion of characteristic elements: if M is a finitely generated torsion $\Lambda(\Gamma)$-module (so we may view $H = 1$), then the Akashi series of M coincides with the $\Lambda(\Gamma)$-characteristic element of M. In this case, we simply write f_M for $\mathrm{Ak}(M)$.

Given a discrete p-primary G-module Y such that $Y^\vee \in \mathfrak{M}_H(G)$, we also denote the Akashi series of $Y^\vee$ by $\mathrm{Ak}(Y)$, by abuse of notation. If Y has finite generalised G-Euler characteristic, then its Akashi series satisfies

$$\chi(G, Y) = |\alpha_Y|_p^{-1},$$

where α_Y is the first non-zero coefficient of $\mathrm{Ak}(Y)$, and the order of vanishing of $\mathrm{Ak}(Y)$ is given by the alternating sum of the $\mathbb{Z}_p$-coranks of the homology groups $H^i(H, Y)^\Gamma$. This generalises the Euler characteristic formula 4.

We have the following comparison formula for the Akashi series:

Theorem 7.3. *Let p be any prime. Let A be an abelian variety defined over a global function field F with* $\mathrm{char}(F) \neq p$. *Let F_∞/F be an admissible p-adic Lie extension. Then we have*

$$\mathrm{Ak}\left(\mathrm{Sel}(A/F_\infty)\right) \mathrm{Ak}\left(A_{p^\infty}(F_\infty)\right)^{-1} = f_{\mathrm{Sel}(A/F^{\mathrm{cyc}})} f^{-1}_{A_{p^\infty}(F^{\mathrm{cyc}})} \prod_{v\in S'} f_{J_v(A/F^{\mathrm{cyc}})}.$$

We point out the striking symmetry in this formula. The Akashi series for the non-commutative extension F_∞/F is determined by its commutative counterpart over F^{cyc}/F, corrected by local terms at the primes in S'.

8 Outline of the Proof of the Main Theorem

We end with a sketch of Theorem 5.1. The proof relies on tools from Galois cohomology. We recall that S denotes a non-empty set of primes of F containing those where A has bad reduction and those that ramify in F_∞/F. Let F_S be the maximal extension of F unramified outside S. The Selmer group fits into the fundamental exact sequence of Galois cohomology:

$$0 \to \mathrm{Sel}(A/F_\infty) \to H^1(G_S(F_\infty), A_{p^\infty}) \to \bigoplus_{v\in S} J_v(A/F_\infty),$$

[1] The term *Akashi series* was introduced by Coates, inspired by the *Tale of Genji*.

where $G_S(F_\infty) = \mathrm{Gal}(F_S/F_\infty)$ and $J_v(A/F_\infty)$ denotes the local cohomology groups.

Thus, to prove that $X(A/F_\infty)$ is finitely generated over $\Lambda(H)$, it suffices, by Nakayama's lemma and duality, to show that the H-invariants $H^i(G_S(F_\infty), A_{p^\infty})^H$ is cofinitely generated over $\mathbb{Z}_p$. The core of the argument is to establish the following key cohomological property.

Theorem 8.1. *The Galois group $G_S(F^{\mathrm{cyc}})$ has p-cohomological dimension 1, that is, $\mathrm{cd}_p(G_S(F^{\mathrm{cyc}})) = 1$.*

This implies that $H^2(G_S(F^{\mathrm{cyc}}), A_{p^\infty}) = 0$. Using the Hochschild–Serre spectral sequence for the group extension $1 \to \mathrm{Gal}(F_S/F_\infty) \to \mathrm{Gal}(F_S/F^{\mathrm{cyc}}) \to H \to 1$, which degenerates due to the properties of an admissible extension, we find an exact sequence:

$$0 \to H^1(H, A_{p^\infty}(F_\infty)) \to H^1(G_S(F^{\mathrm{cyc}}), A_{p^\infty}) \to H^1(G_S(F_\infty), A_{p^\infty})^H \to \dots$$

Since the H-cohomology groups are cofinitely generated over $\mathbb{Z}_p$, the problem reduces to showing that $H^1(G_S(F^{\mathrm{cyc}}), A_{p^\infty})$ is cofinitely generated over $\mathbb{Z}_p$.

Again by Nakayama's Lemma, this is equivalent to proving that the p-torsion subgroup $H^1(G_S(F^{\mathrm{cyc}}), A_{p^\infty})[p]$ is finite. The long exact sequence in Galois cohomology arising from the Kummer sequence $0 \to A_p \to A_{p^\infty} \xrightarrow{p} A_{p^\infty} \to 0$ gives a surjection:

$$H^1(G_S(F^{\mathrm{cyc}}), A_p) \to H^1(G_S(F^{\mathrm{cyc}}), A_{p^\infty})[p].$$

The group on the left, $H^1(G_S(F^{\mathrm{cyc}}), A_p)$, is finite.

This establishes the required finiteness. Thus, we have shown that $\mathrm{Sel}(A/F_\infty)$ is a cofinitely generated $\Lambda(H)$-module. This is equivalent to $X(A/F_\infty)$ being Σ-torsion by [5, Proposition 2.3] for a certain Ore set Σ. The vanishing of the generalised μ-invariant now follows from, for example, the proof of [3, Proposition 3.3 (i)] (see also [9, Lemma 2.7]). This completes the sketch of the proof of Theorem 5.1. □

We note that Theorem 8.1 crucially depends on the fact that F is a global function field. In contrast, if F is a number field, we have $\mathrm{cd}_p(G_S(F^{\mathrm{cyc}})) = 2$.

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